Calculation of fractional age life insurance net premium liability reserve based on α-power death assumption

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Abstract
This article discusses the survival rate of fractional age and the net premium liability reserve for fractional age based on the α-power death hypothesis (specifically divided into cases of paying once a year and paying m times a year), combined with the specific data of the life table with the help of R language and Actuarial software such as crystal ball compares the specific data fitted by the α-power hypothesis with the three traditional hypotheses, and finally concludes that the use of the α-power death hypothesis can improve the accuracy of fitting the fractional age survival rate and the life insurance net premium liability preparation The prediction accuracy of gold, this conclusion will provide a more accurate idea for all insurance companies and social institutions to calculate the fractional age net premium liability reserve.
Keywords: Fractional Age, UDD Hypothesis, Mortality Constant Hypothesis, Hyperbolic Hypothesis , α-power Death Hypothesis, Liability Reserve

Competing Interests:

The authors declare that there is no conflict of interest.

1.Introduction

In life insurance actuarial science, the calculation of actuarial present values used for pricing various insurance products is based on prior information—specifically, mortality tables derived by institutions through establishing target populations and subsequent tracking. However, mortality tables typically only reflect survival and mortality data for whole-number ages, failing to capture survival and mortality patterns at fractional ages. This significantly wastes valuable information about mortality and survival conditions within the intervals between whole-number ages. Consequently, all data pertaining to whole-number ages is only applicable for calculating actuarial present values in scenarios involving premiums or discrete annuity payments. In reality, however, the number of deaths occurring at fractional ages far exceeds those at integer ages. If integer-age data alone is used to directly estimate mortality rates between integer ages, it introduces errors into premium calculations, annuity payouts, and life insurance reserve accruals. This creates unfairness for insurance policyholders. This also impacts insurers' operating costs and profits, often affecting their accounting practices. Consequently, financial reports may fail to accurately reflect the company's operational status, violating the principles of truthfulness and prudence required by accounting standards. This misleads users of insurance financial reports, potentially leading to unforeseeable consequences. In this context, making assumptions and fitting the distribution of survival functions at fractional ages becomes particularly crucial. More accurate assumptions and fitting significantly enhance the prediction accuracy of survival and mortality at fractional ages, thereby optimizing the precision of actuarial present value calculations at these fractional ages.

Research on fractional age assumption distributions has been ongoing among scholars both domestically and internationally, yielding numerous breakthroughs. Traditional, more classical fractional age assumptions include the linear assumption, exponential assumption, and harmonic assumption. These three traditional assumptions essentially involve linear interpolation, exponential interpolation, and hyperbolic interpolation of mortality data at fractional ages. In reality, survival or mortality between fractional ages is discontinuous rather than continuous. Applying these three assumptions to estimate mortality and survival rates at fractional ages, and then using these estimates to calculate annuities, net premiums, and liability reserves, leads to significant computational errors. This results in low accuracy of the calculated data, often rendering the outcomes of these calculations unusable and of limited reference value. In recent years, China's insurance industry has experienced rapid growth. With the evolution of insurance products and heightened public awareness of insurance, demand for insurance products continues to rise. Researchers have consistently pursued optimizations and reforms in fractional age distribution assumption fitting. One notable advancement is the α-power assumption proposed by Jones in 2000. This hypothesis significantly enhances the accuracy of fractional age distribution fits and the precision of actuarial present value calculations. The following discussion begins with an introduction to the fundamental principles of the α-power hypothesis, followed by an intuitive graphical comparison with the distribution results of three traditional hypotheses. The superiority of the α-power mortality hypothesis is demonstrated through numerical comparisons of three error metrics. Finally, the paper discusses the net premium liability reserve for life insurance based on the α-power mortality hypothesis.

For the sake of convenience in the following discussion, the following symbols are introduced:

Symbol annotations

Table 1

symbol	annotations
(x)	A x -year-old person
t	(x) years of survival time (0<t<1)
μ	Death Force
$S(x)$	Survival function
s(x)	Survival density function
$F(x)$	x 's distribution function
$f(x)$	x 's density function
$l(x)$	The number of newborns expected to survive to age 1 year
q_x	Probability of death within the next year
p_x	(x) will live to be $(x+1)$ years old

$\mu(x+t)$	Death Power Function
L	Prospective Loss
$T(x)$	Remaining lifespan for a person of x years
$K(x)$	The remaining lifespan in whole years for a person aged x years
V	Reserve for liabilities
P	Net premiums
A	Actuarial present value of life insurance
a	The present value of a survival annuity
ω	The maximum human age

2.Analysis of α-power death hypothesis and model fitting for three traditional hypotheses

2.1 Model theory

When discussing the value of $l(x)$ at non-integer points, the analysis is typically conducted in segments, generally with one year as a segment. Therefore, the following discussion concerns $l(x+1)$ for any integer $x(x=0,1,2,\ldots,\omega-1)$ and arbitrary $t\left(0<t<1\right)$, given that $l(x)$ and $l(x+1)$ are known.

1.α-power assumptions

The Jones assumptions has the following form（$\alpha \neq 0$）：

$$s(x+t)^{\alpha}=(1-t)\mathrm{s}(x)^{\alpha}+ts(x+1)^{\alpha} \quad (1)$$

$$_{t}q_{x}=1-(1-t+tp_{x}^{\alpha})^{1/\alpha} \quad (2)$$

$$\mu\left(x+t\right)=\frac{\left(1-p_{x}^{\alpha}\right)^{1/\alpha}}{\alpha(1\text{-}t+tp_{x}^{\alpha})} \quad (3)$$

2. Three traditional assumptions

（1）Linear assumption. The linear hypothesis, also known as the UDD hypothesis, essentially involves linear interpolation of the functional distribution across ages. This interpolation method is based on the assumption that $l(x+t)$ takes the following linear form:

$$l\left(x+t\right)=\left(1-t\right)l\left(x\right)+tl\left(x+1\right) \quad (4)$$

Under this assumption, there is:

$$s(x+t)=(1-t)s(x)+ts(x+1) \quad (5)$$

$$_{t}p_{x}=1-tq_{x} \quad (6)$$

$$_{t}q_{x}=tq_{x} \quad (7)$$

$$\mu(x+t)=\frac{q_{x}}{1-tq_{x}} \quad (8)$$

（2） Index assumption. The exponential assumption, also known as the log-linear assumption or constant mortality assumption, posits that $l(x+t)$ takes an exponential form, meaning it can be expressed as ab^{t}. Under this assumption:

$$\ln l(x+t)=(1-t)l(x)+t\ln l(x+1) \quad (9)$$

$$\ln s(x+t)=(1\text{-t})\ln s(x)+t\ln s(x+1) \quad (10)$$

$$_{t}p_{x}=q_{x}^{t} \quad (11)$$

$$_{t}q_{x}=1-(1-q_{x})^{t} \quad (12)$$

$$\mu(x+t)=-\log p_{x} \quad (13)$$

（3） Hyperbolic assumption. The hyperbolic assumption, also known as the harmonic assumption or Balducci's assumption, posits that $l(x+t)$ takes the form of a hyperbola, meaning it can be expressed as $(a+bt)^{-1}$:

$$\frac{1}{l(x+t)}=\frac{1-t}{l(x)}+\frac{t}{l(x+t)} \quad (14)$$

Under this assumption, there is:

$$\frac{1}{s(x+t)}=\frac{1-t}{s(x)}+\frac{t}{s(x+1)} \quad (15)$$

$$_{t}p_{x}=\frac{p_{x}}{1-(1-t)q_{x}} \quad (16)$$

$$_{t}q_{x}=\frac{tq_{x}}{1-(1-t)q_{x}} \quad (17)$$

$$\mu(x+t)=\frac{q_{x}}{1-(1-t)q_{x}} \quad (18)$$

2.2 Error analysis of the precision of four hypothesis fits using R

To compare the fit of four assumptions, this paper developed a program in R software based on data from the Chinese Life Insurance Industry Experience Tables (Male Table, Non-Pension Business, 2010–2013). Four traditional assumption models were constructed, and interpolation was performed for each assumption using even-aged data as known values. The interpolated results for odd-aged data were then compared and analyzed against the actual values from the life tables.

To ensure clear numerical results for the four hypotheses, facilitating comparative analysis and assessment of fitting accuracy, this paper also employs three commonly used criteria for evaluating error magnitude to assess the interpolation performance of these different hypotheses.

These criteria are:

Root mean square error: $RMSE = \sqrt{\frac{\sum_{i=1}^{n}(A_i - B_i)^2}{n}}$ (19)

Maximum absolute error: $MAE = max|\ A_i - B_i|$ $\quad n = 1,2,3\ldots\ldots$(20)

Mean absolute error: $AAE = \frac{\sum_{i=1}^{n}|\ A_i - B_i|}{n}$ (21)

Where n is the number of fractional age points, A_i is the actual survival function value, and B_i is the predicted survival function value under different interpolation assumptions.

Evaluation criteria

Table 2

Evaluation Criteria	Relationship between numerical values and model accuracy	Measurement criteria
RMSE	The smaller the value, the more accurate the model	Assess the overall model accuracy
MAE	The smaller the value, the more accurate the model	Measuring local model accuracy
AAE	The smaller the value, the more accurate the model	Assess the overall model accuracy

Chinese life insurance industry experience life tables (2010–2013)

Table 3

age	Non-Pension Business Form 1	Non-Pension Business Form 2	Pension Business Form

	Male （CL1）	Female （CL2）	Male（CL3）	Female （CL4）	Male（CL5）	Female （CL6）
0	0.000867	0.00062	0.00062	0.000455	0.000566	0.000453
1	0.000615	0.000456	0.000465	0.000324	0.000386	0.000289
2	0.000445	0.000337	0.000353	0.000236	0.000268	0.000184
3	0.000339	0.000256	0.000278	0.00018	0.000196	0.000124
4	0.00028	0.000203	0.000229	0.000149	0.000158	0.000095
5	0.000251	0.00017	0.0002	0.000131	0.000141	0.000084

1. Interpolation results

Interpolation results for four assumptions

Table 4

s(x)	1	3	···	49	51	···
α -power assumption	0.99897	0.998022	···	0.953641	0.945865	···
UDD assumption	0.999259	0.999608	···	0.996282	0.995549	···
Linear assumption	0.999259	0.999608	···	0.996289	0.995559	···
Hyperbolic assumption	0.999259	0.999735	···	0.996296	0.995569	···

The following discussion pertains to the values of α:

According to Jones' α-power hypothesis at integer age points: The left limit and right limit of the mortality function are equal, yielding:

$$\frac{1-p_x^{\alpha_x}}{\alpha_x p_x^{\alpha_x}} = \frac{1-p_{x+1}^{\alpha_{x+1}}}{\alpha_{x+1}} \qquad x = 0,1,2\ldots\ldots\omega-2 \,(21)$$

Assuming further that the logarithmic function of the force of death achieves a minimum when the sum of the squares of the differences between the left and right derivatives at all integer age points is minimized, we can obtain the values of α at different ages.

$$\log\mu_{x+t}|_{t=1} - \frac{\partial}{\partial t}\log\mu_{x+t+1}|_{t=0})^2 = \sum_{x=0}^{\omega-2}(p_x^{\alpha_x} + p_{x+1}^{\alpha_{x+1}} \text{-}2)^2 \,(22)$$

For the sake of discussion, this paper makes the following assumptions:

Assuming that α remains constant across different age groups and takes specific values of -10000, -1000, -100, -10, -0.00001, and 0.00001 when input into the model, interpolation calculations for odd ages yield the following results:

Interpolation results for different α values

α \ s(x)	1	3	···	49	51	···
-10000	0.999174	0.998098	···	0.953975	0.94623	···
-1000	0.999174	0.998098	···	0.953975	0.94623	···

-100	0.999174	0.998098	···	0.953975	0.94623	···
-10	0.999174	0.998098	···	0.953975	0.94623	···
-0.00001	0.99897	0.998022	···	0.953641	0.945865	···
0.00001	0.99897	0.998022	···	0.953641	0.945865	···

Table 5

To obtain a more accurate α fitting value, the sum of squares of the left and right derivatives at all integer age points is used as a parameter to evaluate the quality of the fit.

Sum of squares of differences between left and right derivatives for different α

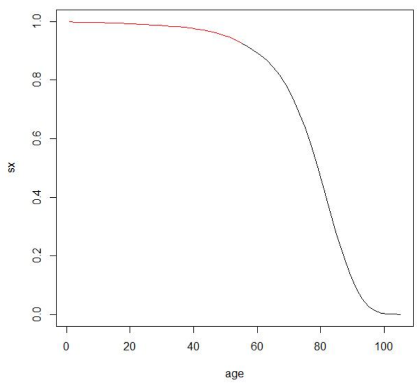
Table 6

α	Sum of the squares of the left and right derivatives
-10000	1.5539E+302
-1000	5.7993E+211
-100	7.7312E+252
-10	3.86948E+55
-0.00001	3.53183E-12
0.00001	3.53183E-12

It can be seen that the best fit is achieved when α is set to -0.00001. Therefore, all instances of α in this study are assumed to be -0.00001.

2. Visualization

（1） α -power assumption



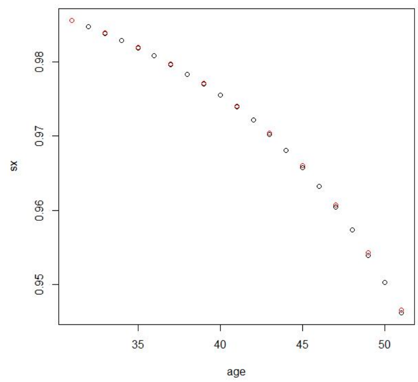
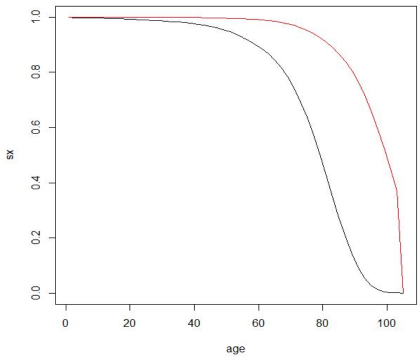


Figure 1 comparison　　Figure 2 Local comparison

（2）UDD assumption



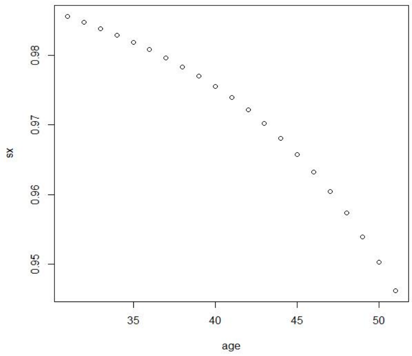
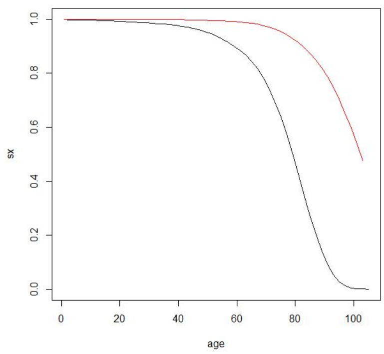


Figure 3 comparison　Figure 4 Local comparison

（3）Linear assumption



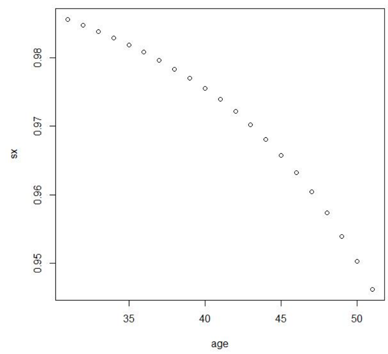
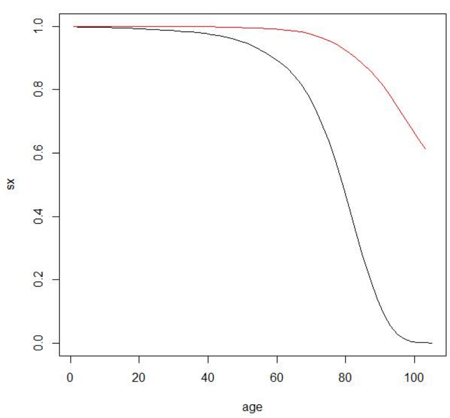


Figure 5 comparison　Figure 6 Local comparison

（4）Hyperbolic assumption



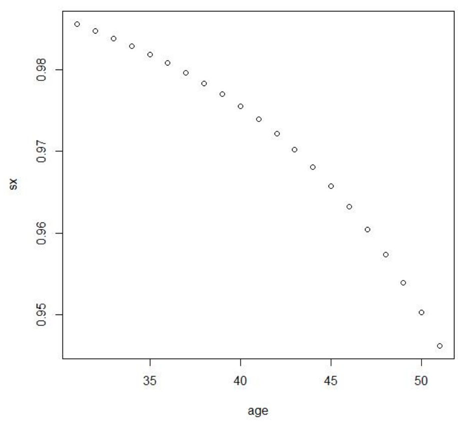
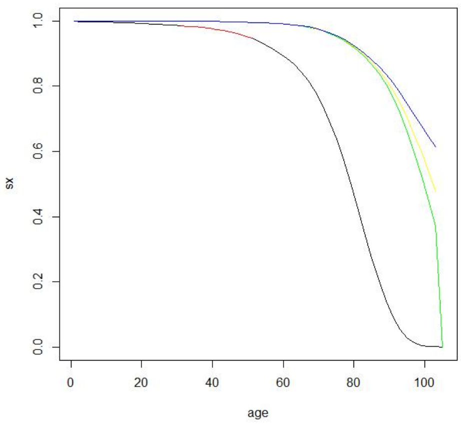


Figure 7 comparison　Figure 8 Local comparison

（5）Overall comparative analysis



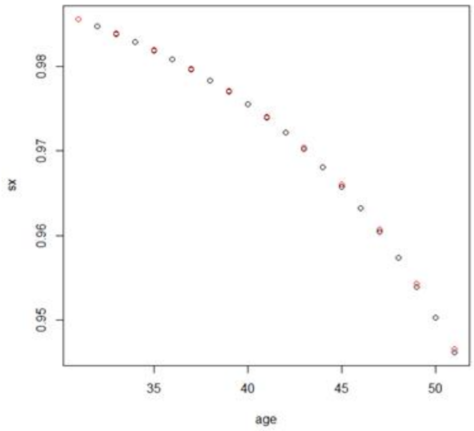


Figure 9 comparison Figure 10 Local comparison

The ten figures above provide a visual representation of the fitting performance for the α-power hypothesis and the three traditional hypotheses. First, analyzing the goodness-of-fit for the three traditional hypotheses: all three hypotheses exhibit relatively ideal fitting in the early age range but perform poorly in the middle and late age ranges. Among them, the hyperbolic hypothesis performs the worst, with its fitting in the middle and late stages failing to adequately reflect the true value levels. It is evident that among the three traditional assumptions for fitting survival functions at fractional ages, the constant mortality assumption and uniform distribution assumption perform moderately, while the hyperbolic assumption is the least effective at reflecting reality. When comparing the fitting results with the α-power assumption, it becomes clear that the α-power assumption better fits survival patterns at fractional ages than the three traditional assumptions. It accurately reflects survival and mortality status at fractional ages throughout the entire human lifespan.

In practical applications, the α-power mortality assumption should be prioritized for fitting unknown fractional ages. Alternatively, the constant mortality rate or uniform distribution assumptions may be considered for approximating fractional ages during the early stages of human survival. The hyperbolic assumption should only be employed as a last resort.

3. Calculation results for three evaluation criteria

Calculation results for three evaluation criteria

Table 6

Different assumptions / Evaluation criteria	α-power	UDD	Linear	Hyperbolic
RMSE	0.000320	0.000371	2.48712E-05	0.001082

MAE	6.92718E-06	0.000127	0.000126	0.000128
AAE	0.000914	0.006474	0.002495	0.012227

The above values yield a result consistent with that observed from the graphical representation, where:

RMSE：RMSE1<RMSE3<RMSE2<MSE4

MAE：MAE1<MAE3<MAE2<MAE4

AAE：AAE1<AAE3<AAE2<AAE4

Based on the root mean square error (RMSE1 < RMSE3 < RMSE2 < MSE4) and average absolute error (AAE1 < AAE3 < AAE2 < AAE4), the following conclusion can be drawn: Overall, the α-power mortality distribution hypothesis among the four hypotheses provides the best fit, with the smallest discrepancy between predicted and actual values, resulting in the lowest root mean square error and average absolute error values. The uniform distribution hypothesis fits less well than the α-power mortality distribution hypothesis. Compared to the constant mortality rate and hyperbolic hypotheses, it outperforms the hyperbolic hypothesis but underperforms the constant mortality rate hypothesis. The mean square error and mean absolute error values of the constant mortality assumption fall between those of the α-power mortality distribution and hyperbolic assumptions. This precisely indicates that, compared to the best-fitting α-power mortality assumption and the worst-fitting hyperbolic assumption, the uniform distribution and constant mortality assumptions exhibit intermediate fitting performance among the four traditional assumptions.

From a local perspective, since the maximum absolute errors follow the order MAE1 < MAE3 < MAE2 < MAE4, the α-power mortality distribution hypothesis demonstrates the best local fitting performance among the four hypotheses. Among the three traditional hypotheses, the uniform distribution hypothesis and constant mortality hypothesis exhibit the next best local fitting performance, while the hyperbolic hypothesis shows the poorest local fitting performance.

The results from calculating the root mean square error, maximum absolute error, and mean absolute error clearly demonstrate from a data perspective that, whether viewed holistically or locally, the α-power hypothesis among the three traditional hypotheses exhibits the best fit. The uniformly distributed hypothesis and constant mortality hypothesis yield the next best results, while the hyperbolic hypothesis produces the least satisfactory fit. This demonstrates that the α-power hypothesis significantly enhances the accuracy of predicting mortality or survival outcomes for individuals with unknown scores, establishing it as an excellent hypothesis-fitting method.

4．Comparative analysis of four hypothesis fitting scenarios

When considering only the fitting performance of the three traditional assumptions, the constant mortality assumption yields the most predictive fitting accuracy, followed by the uniform distribution assumption, while the hyperbolic assumption is the least desirable. This is specifically reflected in the following:

First, the intuitive plots of the three assumptions clearly show that the constant mortality assumption provides a relatively suitable and reference-worthy fit for the survival function at fractional ages. However, it also visually reveals its inadequacy in fitting the middle and later stages of human age. Next, examining the three evaluation criteria, the constant mortality assumption yields the smallest root mean square error (RMSE) at RMSE=2.48712E-05. Thus, it can be concluded that the constant mortality assumption achieves a relatively high level of fitting precision for the survival function at fractional ages, both in terms of visual graph fitting and overall RMSE. Furthermore, its maximum absolute error is MAE=0.000126, demonstrating its accuracy in local fitting. Considering the constant mortality assumption's average absolute error (AAE) of 0.002495, this value further indicates that the constant mortality assumption maintains a high level of overall fitting precision.

Analyzing Figures 3 and 4, which present the fitting results for the uniform distribution hypothesis, we observe that the uniform distribution hypothesis initially demonstrates relatively good fitting performance at the beginning of the curve. However, as age increases, the fitting accuracy of the uniform distribution hypothesis gradually deteriorates. Among the three evaluation criteria for fitting, the uniform distribution hypothesis exhibits a root mean square error (RMSE) of 0.000371, which is at an intermediate level among the three traditional assumptions. This leads to the conclusion that, overall, the uniform distribution assumption exhibits poorer fitting precision than the constant mortality assumption but performs better than the hyperbolic assumption. The mean absolute error (MAE) of the uniform distribution assumption is 0.000127, also falls within the middle range among the three traditional assumptions. This indicates that the uniform distribution assumption exhibits average performance in local fitting accuracy. Furthermore, its average absolute error (AAE) of 0.0064741 confirms that the overall fitting accuracy of the uniform distribution assumption is moderately positioned among the three traditional assumptions.

Finally, the visual comparison of predicted and actual values for the hyperbolic assumption reveals consistently poor performance. Its root mean square error (RMSE) of 0.001082 indicating suboptimal overall fitting. The maximum absolute error (MAE) of 0.000128 suggests the hyperbolic assumption also performs poorly in local fitting of the fractional age survival function. The average absolute error (AAE) of 0.012227 further confirms the hyperbolic assumption's inferior fitting capability for fractional age survival functions. In summary, whether assessed visually or through the three specific evaluation criteria, the constant mortality assumption yields the best fit, followed by the uniform distribution assumption, with the hyperbolic assumption producing the poorest fit. Therefore, in practical applications, the constant mortality assumption or uniform distribution assumption should be prioritized, with the hyperbolic assumption selected only as a last resort.

Considering the fitted model analysis incorporating the α-power assumption, the visual plot clearly demonstrates that the α-power hypothesis better fits the survival or mortality distribution across the entire lifespan compared to the three traditional distribution assumptions. From another perspective, examining the three error metrics, the α-power hypothesis achieves the smallest values for both the root mean square error (RMSE) and the maximum absolute error (MAE) among the four fitted models. Therefore, it can be concluded that the α-power hypothesis overcomes the limitations of the three traditional assumptions in fitting survival or mortality data during the middle and late stages of human life. maximum absolute error (MAE), or average absolute error (AAE), the α-power hypothesis yields the smallest values among the four fitted models. Therefore, it can be concluded that the α-power hypothesis overcomes the shortcomings of the three traditional hypotheses in terms of inaccurate fitting during the middle and late stages of life, thereby improving the precision of predicting unknown survival or death at fractional ages. It is an excellent fractional age fitting method.

As demonstrated by the liability reserve calculation formula below, insurance companies base their fractional age reserve accruals on fractional age survival rates. Since the α-power hypothesis effectively models fractional age survival patterns, its application in calculating fractional age liability reserves enhances the precision of reserve amounts and yields the most meaningful reference data. The following section specifically discusses net premium liability reserves for term life insurance based on the α-power mortality assumption.

3. Net Premium Liability for Fractional-age Term Life Insurance under the α-Power Mortality Assumption

This section primarily discusses the formula for calculating the net premium liability reserve that insurance companies should set aside for policyholders at fractional ages under the α-power assumption. For simplicity, the following three scenarios assume the policyholder has purchased whole life insurance. Similar methods can be applied to calculate and analyze other types of life insurance policies.

3.1 Lifetime life insurance liability reserve for death benefits paid at the end of the year with annual premium payments

Assuming premiums are paid annually and the insurance benefit is paid at the end of the year of the insured's death, with the death benefit at the end of the $j+1$ policy year being b_{j+1}, consider (x) whole life insurance policy with j. The net premium collected by the insurer at the end of the policy year is π_j, paid at the beginning of each policy year. Let the liability reserve at the end of the $h+t$ year be ${}_{h+t}V_x$, which is the conditional mathematical expectation of the prospective loss ${}_{h+t}L$ at age $h+t$, where g is the integer age reached by the person at age (x) at death, and $t(0<t<1)$ is the fraction of the year exceeding the integer age.

When $k(x) \le h-1$, ${}_{h+t}L=0$. When $k(x)=h$, ${}_{h+t}L=v^{1-t}b_{k(x)+1}$. When $k(x) \ge h+1$, ${}_{h+t}L=v^{[K(x)+1-h-t]}b_{K(x)+1}-\sum_{j=h+1}^{K(x)} v^{j-h-t}\pi_j$.

Then (x) the liability reserve at time $h+t$ is ${}_{h+t}V=E({}_{h+t}L|\ T(x) \ge h+t)$, and ${}_{h+t}V=v^{1-t}b_{h+1\ 1-t}q_{x+h+t}+v^{1-t}{}_{h+1}V_{1-t}p_{x+h+t}$, finally we have:

$$v^t{}_tp_{x+h\ h+t}V=\left({}_hV+\pi_h\right)\frac{{}_{t|1-t}q_{x+h}}{q_{x+h}}+{}_{h+1}Vvp_{x+h}(1-\frac{{}_{t|1-t}q_{x+h}}{q_{x+h}})\ (23)$$

This indicates that the actuarial present value of the liability reserve at time is an interpolation between the actuarial present values of the liability reserve at times $h+t$ and $h+1$.

Substitute the formula derived earlier for the age-specific mortality rate and survival rate under the α-power assumption into the aforementioned liability reserve calculation formula.

$$v^t{}_tp_{x+h\ h+t}V=\left({}_hV+\pi_h\right)\frac{(1-t+tp_x{}^{\alpha})^{1/\alpha}-p_{x+t}}{q_{x+t}}+{}_{\mathrm{h+1}}Vvp_{x+h}\frac{(1-t+tp_x{}^{\alpha})^{1/\alpha}-p_{x+t}}{q_{x+t}}\ (24)$$

3.2 Lifetime life insurance liability reserve for death benefits paid at the end of the year with m annual premium payments

For the sake of simplicity, we will illustrate the calculation of life insurance reserve for multiple annual premiums using the example of two annual payments. Assume the policyholder makes two payments per year, each amounting to π_h. The death benefit is payable at the end of the policy year. The reserve at time $h+t$ is denoted as ${}_{h+t}V^{(2)}$.

For $0<t\le\frac{1}{2}$, the forward formula yields

$${}_{h+t}V^{(2)}=v^{1-t}b_{h+1\ 1-t}q_{x+h+t}+v^{1-t}{}_{h+1}V^{(2)}{}_{1-t}q_{x+h+t}-\frac{\pi_h}{2}v^{0.5-t}{}_{0.5-t}q_{x+h+t}\quad(25)$$

The first term on the right-hand side of the above equation represents the actuarial present value of the insurance coverage amount for the current year. The second term denotes the actuarial present value of the liability reserve at year-end. The third term signifies the actuarial present value of the premium paid at time $h+\frac{1}{2}$ during the year.

Multiplying both sides of the above equation by $v^t{}_t p_{x+h}$ yields

$$v^t{}_t p_{x+h\;h+t}V^{(2)} = vb_{h+1\,t|1-t}q_{x+h} + v_{h+1}V^{(2)}p_{x+h} - \frac{\pi_h}{2}v^{0.5}{}_{0.5}q_{x+h+t} \quad (26)$$

For the year-end net reserves, the recursive formula is as follows:

$$_hV^{(2)} = b_{h+1}vq_{x+h} + {}_{h+1}V^{(2)}vp_{x+h} - \frac{\pi_h}{2}\left(1 + v^{0.5}{}_{0.5}p_{x+h}\right) \quad (27)$$

Then we have:

$$b_{h+1}v = \frac{_hV^{(2)} - {}_{h+1}V^{(2)}vp_{x+h} + \frac{\pi_h}{2}\left(1 + v^{0.5}{}_{0.5}p_{x+h}\right)}{q_{x+h}} \quad (28)$$

Substitute the above expression into the equation:

$$v^t{}_t p_{x+h\;h+t}V^{(2)} = ({}_hV^{(2)} + \frac{\pi_h}{2})\frac{_{t|1-t}q_{x+h}}{q_{x+h}} + [{}_{h+1}V^{(2)}vp_{x+h} - \frac{\pi_h}{2}v^{0.5}{}_{0.5}p_{x+h}](1 - \frac{_{t|1-t}q_{x+h}}{q_{x+h}})$$

(29)

The above formula indicates that the actuarial present value of the liability reserve at time $h+t$ is equal to h, which is a nonlinear interpolation between the opening liability reserve at time h and the present value of the closing liability reserve at time $h+1$.

Under the assumption that the mortality distribution is an α-power hypothesis, there is:

$$\begin{aligned} v^t{}_t p_{x+h\;h+t}V^{(2)} &= ({}_hV^{(2)} + \frac{\pi_h}{2})\ \frac{(1-t+tp_x{}^{\alpha})^{1/\alpha}) - p_{x+t}}{q_{x+t}} \\ &+ {}_{h+1}V^{(2)}vp_{x+h} - \frac{\pi_h}{2}v^{0.5}{}_{0.5}p_{x+h}\ \frac{(1-t+tp_x{}^{\alpha})^{1/\alpha}) - p_{x+t}}{q_{x+t}} \end{aligned} \quad (30)$$

For the second half of the year, similar to what is available:

$$v^t{}_t p_{x+h\;h+t}V^{(2)} = [{}_hV^{(2)} + \left(1 + v^{0.5}{}_{0.5}p_{x+h}\right)\frac{\pi_h}{2}]\frac{_{t|1-t}q_{x+h}}{q_{x+h}} + {}_{h+1}V^{(2)}vp_{x+h}\,(1 - \frac{_{t|1-t}q_{x+h}}{q_{x+h}})$$

(31)

Similarly, under the assumption that the death distribution is an α-power hypothesis, there is:

$$v^t {}_t p_{x+h} {}_{h+t}V^{(2)} = [{}_hV^{(2)} + \left(1+ v^{0.5} {}_{0.5}p_{x+h}\right)\frac{\pi_h}{2}]\frac{(1-t+tp_x{}^{\alpha})^{1/\alpha} - p_{x+t}}{q_{x+t}} + {}_{h+1}V^{(2)} v p_{x+h} \frac{(1-t+tp_x{}^{\alpha})^{1/\alpha} - p_{x+t}}{q_{x+t}} \quad (32)$$

For general life insurance products with m annual premium payments, similar treatment can be applied.

As seen from the formula for fractional age reserves above, the reserve calculation is influenced by both the predetermined interest rate and the fractional age survival rates. Since interest rates are beyond an insurer's control, the decisive factor affecting fractional age reserve calculations is the fractional age survival rates. Corresponding case studies will be analyzed below. The comparative analysis of the α-power hypothesis model and three traditional distribution assumptions in Section 2 demonstrates that employing the α-power model enhances the accuracy of fractional age survival rate calculations. Consequently, using the α-power model to fit fractional ages improves the precision of fractional age reserve calculations.

3.3 Case analysis

In this section, fractional ages between 0-1 and 50-51 will be selected. Using data from Table 1 (Male) of the Non-Pension Business section in the Chinese Life Insurance Industry Experience Tables (2010-2013), specific net premium liability reserve values for life insurance will be calculated based on an annual premium payment method. To highlight the advantage of the α-power hypothesis in fractional age fitting, the uniform distribution assumption—the most common among three traditional assumptions—is selected for comparison with the α-power hypothesis. Considering that different interest rate levels may affect calculation results, two distinct interest rates are chosen to eliminate such impacts. The assumed interest rates are 0.025 and 0.05, $L_0 = 1000000$, respectively, with the benefit amount set at unit 1. For policies with multiple annual premiums, calculations can be performed as follows.

Calculation results for net premium liability reserves for fractional-age groups aged 0-1 years

Table 7

age		UDD	α-power
i=0.025	0.3	0.004516	0.004518
	0.5	0.004368	0.004371
	0.8	0.004143	0.004144
i=0.05	0.3	0.001558	0.001577
	0.5	0.001329	0.001362

	0.8	0.001085	0.001086

Calculation Results for Net Premium Liability Reserves for Age Groups 50-51

Table 8

age		UDD	α -power
i=0.025	50.3	0.421505	0.421507
	50.5	0.423112	0.423118
	50.8	0.425530	0.425533
i=0.05	50.3	0.260475	0.262530
	50.5	0.263254	0.264499
	50.8	0.267477	0.267480

Analysis of Table 7 and Table 8 reveals that:

First, results derived under the α-power mortality assumption yield larger values than those under the uniform distribution assumption. Therefore, if liability reserves are calculated based on the traditional assumption, it would lead to insufficient reserve provisions by insurance companies to cover future claims payments, thereby impacting their financial operations.

Second, although different interest rates appear to significantly impact the calculation of liability reserves, the numerical difference between results under the two assumptions is negligible. Therefore, varying interest rates should have no effect on liability reserve calculations. The same principle applies to different ages. Although different age groups have a substantial impact on the results, their effect on the difference between the two calculation outcomes is negligible. Consequently, the influence of interest rate factors and age group factors on the calculation of technical reserves can be disregarded.

From the above analysis, we observe that selecting different hypothetical models for fractional-age liability reserve calculations yields varying outcomes. Employing a more precise hypothesis—specifically the α-power hypothesis in this study—yields a more accurate fit and a result closer to reality. Consequently, using the α-power hypothesis to calculate fractional-age net premium liability reserves for life insurance reduces operational risks for insurance companies.

4. Conclusion

The provisioning of life insurance reserves relates to an insurer's asset-liability management and impacts its financial reporting. Uncertain survival status at fractional ages also influences decisions made by insurance operators and judgments made by users of financial reports. Enhancing the accuracy of fractional-age survival and mortality rate fitting can mitigate these issues to a certain extent.

Regarding the superiority of fractional age fitting assumptions, the proportion of deaths occurring at specific fractional ages exceeds that of deaths occurring precisely at integer ages. Therefore, studying fractional age mortality and survival distributions holds greater theoretical significance than relying solely on integer age data. Empirical evidence indicates that the three traditional distribution assumptions exhibit significant limitations in fitting fractional ages. Analysis in this paper demonstrates that employing Jones' α-power hypothesis for fitting fractional-age mortality and survival yields substantially improved fitting precision compared to the traditional assumptions. This superiority is evident both visually and through three error evaluation metrics. or from the analysis of specific numerical fitting results, the conclusion remains consistent: the α-power mortality assumption yields more precise fitting for unknown fractional-age mortality or survival data compared to the three traditional distribution assumptions. This finding provides insurers with a reference approach for enhancing asset-liability management.

On the other hand, examining the formula for calculating life insurance policy reserves directly reveals—through the reasoning and case analysis presented in the latter part of this paper—that the calculation relies on survival rates and mortality rates based on fractional age. Therefore, the precision of these rates directly impacts the accuracy of reserve accrual. In accounting practices, policy reserves constitute a critical component of financial reporting, impacting disclosures in financial statements that influence both internal and external users. For internal stakeholders, an insurer's financial reports serve as vital decision-making references for management, forming the foundation for future development. For external users of financial information, these reports provide direct basis for judgment. Consequently, insurers must enhance the accuracy of reserve calculations to optimize financial disclosure.

This paper demonstrates that under the α-power hypothesis proposed by Jones, the goodness-of-fit for unknown fractional-age survival rates outperforms three traditional hypothetical distributions. This approach improves the precision of fractional-age mortality rate fitting and reserve calculation results to a certain extent. This conclusion provides a reference method for insurance companies to enhance their actuarial management capabilities and address the issue of accruing fractional-age reserves in company accounting.

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